Kuwait University

Department of Mathematics — First In-term Test, 7 April 2011

Math 101: Calculus I — Duration 90 minutes

Answer all of the following questions.
Calculators and mobile telephones are not allowed.

1. [6 pts.] Compute the following limits if they exist. If a limit does not exist, clearly state why.

   (a) $\lim_{x\to 4}\left(\frac{1}{x-4}-\frac{3}{x^2-5x+4}\right)$

   (b) $\lim_{x\to\infty}\frac{x\sin x}{x^2+1}$

2. [3 pts.] Show that the equation $2x+\sin x=5$ has a solution in the interval $[\pi/2,\pi]$.

3. [4 pts.] Find and classify (removable, jump, or infinite) the discontinuities of the function

$$f(x)=\frac{(x-2)|x-1|}{(x-1)(x^2-4)}.$$

4. [3 pts.] The following limit represents $f'(a)$ for a function f and a number a. Identify f and a, and use this information to compute the limit.

$$\lim_{x\to -1}\frac{x^{100}-1}{x+1}$$

5. [6 pts.] Find the derivative of the following functions.

   (a) $f(x)=x\left[x+(x+\cos^2 x)^3\right]$

   (b) $g(t)=\sqrt{\frac{t}{t^2+4}}$

6. [3 pts.] At what points on the curve

$$y=\frac{x-1}{x+1}$$

is the tangent line parallel to the line $x-2y=2$?

1. (a)
$$\lim_{x\to 4}\left(\frac{1}{x-4}-\frac{3}{x^2-5x+4}\right)$$
$$= \lim_{x\to 4}\left[\frac{1}{x-4}-\frac{3}{(x-1)(x-4)}\right]=\lim_{x\to 4}\frac{(x-1)-3}{(x-1)(x-4)}=\lim_{x\to 4}\frac{1}{x-1}=\frac{1}{3}.$$

   (b) $-1 \leqslant \sin x \leqslant 1$ for all x. Thus
$$-\frac{x}{x^2+1}\leqslant\frac{x\sin x}{x^2+1}\leqslant\frac{x}{x^2+1}\qquad\text{for all } x\geqslant 0.$$
   Furthermore,
$$\lim_{x\to\infty}\frac{x}{x^2+1}=\lim_{x\to\infty}\frac{1}{x(1+1/x^2)}=0.$$
   So by the Squeeze Theorem,
$$\lim_{x\to\infty}\frac{x\sin x}{x^2+1}=0.$$

2. Let $f(x) = 2x + \sin x$. Then f is continuous on $[\pi/2, \pi]$, $f(\pi/2) = \pi + 1 < 5$, and $f(\pi) = 2\pi > 5$. So by the Intermediate Value Theorem, the equation $f(x) = 5$ has a solution in the interval $(\pi/2, \pi)$.

3. The function is defined everywhere except where its denominator is 0, i.e. for all $x \neq 1, 2, -2$, and simplifies to
$$f(x)=\begin{cases}1/(x+2) & \text{for } x>1, x\neq 2\\ -1/(x+2) & \text{for } x<1, x\neq -2.\end{cases}$$
   Thus, f is piecewise rational, and continuous everywhere except at 1, 2, and -2. Since
$$\lim_{x\to 1^-}f(x)=\lim_{x\to 1}\frac{-1}{x+2}=-\frac{1}{3}\quad\neq\quad\lim_{x\to 1^+}f(x)=\lim_{x\to 1}\frac{1}{x+2}=\frac{1}{3},$$
   f has a jump discontinuity at 1. Since
$$\lim_{x\to 2}f(x)=\lim_{x\to 2}\frac{1}{x+2}=\frac{1}{4},$$
   f has a removable discontinuity at 2. Since
$$\lim_{x\to -2^\pm}f(x)=\lim_{x\to -2^\pm}\frac{-1}{x+2}=\mp\infty,$$
   f has an infinite discontinuity at -2.

4. The obvious combination of f and a is $f(x) = x^{100}$ and $a = -1$. This gives

$$\lim_{x\to-1} \frac{x^{100}-1}{x+1} = f'(a) = 100x^{99}\Big|_{x=a} = 100(-1)^{99} = -100.$$

5. (a) $f(x) = x^2 + x(x + \cos^2 x)^3$. Hence by the product rule,

$$f'(x) = 2x + (x+\cos^2 x)^3 + x\frac{d}{dx}(x+\cos^2 x)^3.$$

Then by the chain rule,

$$\begin{aligned} & f'(x) \\ = \;& 2x + (x+\cos^2 x)^3 + x\,3\,(x+\cos^2 x)^2 \frac{d}{dx}(x+\cos^2 x) \\ = \;& 2x + (x+\cos^2 x)^3 + x\,3\,(x+\cos^2 x)^2 \left(1 + 2\cos x \frac{d}{dx}\cos x\right) \\ = \;& 2x + (x+\cos^2 x)^3 + x\,3\,(x+\cos^2 x)^2 \,[1 + 2\cos x\,(-\sin x)] \\ = \;& 2x + (4x + \cos^2 x - 6x\cos x \sin x)(x+\cos^2 x)^2 \end{aligned}$$

(b) $g(t) = \left[t/(t^2+4)\right]^{1/2}$. Hence by the chain rule,

$$g'(t) = \frac{1}{2}\left(\frac{t}{t^2+4}\right)^{-1/2} \frac{d}{dt}\left(\frac{t}{t^2+4}\right).$$

Then by the quotient rule,

$$\begin{aligned} g'(t) &= \frac{1}{2}\left(\frac{t}{t^2+4}\right)^{-1/2}\left[\frac{(t^2+4)-t(2t)}{(t^2+4)^2}\right] \\ &= \tfrac{1}{2}(4-t^2)t^{-1/2}(t^2+4)^{-3/2}. \end{aligned}$$

6. The slope of the curve is

$$\frac{dy}{dx} = \frac{(x+1)-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}.$$

The slope of the given line is $1/2$. Thus, a tangent line to the curve will be parallel to the given line when

$$\frac{2}{(x+1)^2} = \frac{1}{2}$$

$\Longrightarrow$

$$(x+1)^2 = 4 \quad\Longrightarrow\quad x+1 = \pm 2 \quad\Longrightarrow\quad x = 1, -3.$$

If $x = 1$ then $y = (1-1)/(1+1) = 0$.
If $x = -3$ then $y = (-3-1)/(-3+1) = (-4)/(-2) = 2$.
Answer: The tangent line to the curve is parallel to the given line at the points $(1,0)$ and $(-3,2)$.